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2013.07

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

Multiple Solutions in Systems of Functional Differential Equations

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January 22, 2013

Abstract

This paper proposes conditions for the existence and uniqueness of solutions to systems of linear differential or algebraic equations with delays or advances, in which some variables may be non predetermined. The obtained conditions represent the counterpart of the Blanchard and Kahn conditions for the considered functional equations.
JEL Classification: C61, C62, E13
Keywords: Delay Differential Equations, Advance Differential Equations, Existence, Indeterminacy

1 Introduction

A common feature in many dynamic models in economics hinges on the fact that the initial value of some variables is unknown. Moreover, certain asymptotic properties, and notably the convergence toward a steady state, should be taken into account. Mathematically, these are boundary value problems. The analytical resolution method consists in projecting the trajectory onto the stable eigenspace of the dynamic system. By comparing the dimensions of the space of the non-predetermined variables and those of the unstable eigenspace, one can deduce the properties for the existence and determinacy of a solution to the system being considered (Blanchard and Khan, 1980, Buiter, 1984). Equilibrium is said to be indeterminate when there is more than one solution, and sunspot fluctuations may appear (Azariadis, 1981, Benhabib and Farmer, 1999). However, the mathematical theorems that characterize these properties were only established for systems of finite dimensions comprising ordinary differential equations (ODEs) or difference equations. In this paper, we generalize these theorems to include some systems of delay or advanced differential equations (DDEs or ADEs).

As Burger (1956) has pointed out, many dynamic systems in economics can be written in the form of DDE. Since then, DDEs have been used in economic demography, vintage capital, time-to-build and monetary policy literatures (see, Boucekkine et al. 2004, for an excellent survey of the use of DDEs in economics). However, for want of a theorem, up until now, authors have had to confine their work to very specific cases where the stability properties of the dynamics may be proven[1]. Alternatively, they could use numerical methods or other mathematical tools, and most notably optimal control with the Hamilton-Jacobi-Bellman

[1] See, among others, Gray and Turnovsky (1979), Benhabib (2004), Boucekkine et al. (2005), Bambi (2008), Augeraud-Véron and Bambi (2011), d'Albis et al. (2012).

equation[2].

DDE systems, which are characterized by a stable manifold of infinite dimensions, have generated an abundance of mathematical literature (see the textbooks by Bellman and Cooke, 1963, and Diekmann et al. 1995). However, the existing theorems are only valid for systems where all the variables are predetermined and defined as continuous function. We extend these theorems to cases where some variables are non-predetermined – their past values are given but their value when the system is initiated is unknown – and to cases where some predetermined variables are discontinuous. To do so, we have studied in a previous paper (d'Albis, Augeraud-Véron and Hupkes, 2012) the properties of an operator that acts on a multivalued space. In the present paper, we use the results we obtained to rewrite the spectral projection formula according to the initial conditions and the jump made by non-predetermined variables. We set the projection on the unstable manifold to zero and deduce the magnitude of the jump that nullifies the projection on the unstable manifold. The spectral projection formula then enables us to establish the conditions for the existence and uniqueness of a solution. Most notably, we prove that it is possible to come to a conclusion by comparing the dimensions of the space of the unknown initial conditions and those of the unstable eigenspace. Our results also apply to systems of algebraic equations with delays, if their nth derivative is a DDE. In this case, the constraints imposed by such equations must be taken into account in the conditions for existence and uniqueness.

Our conditions can be extended to differential equation with advances. Systems of ADEs are more similar to ODE systems as they have a stable eigenspace of finite dimensions. We demonstrate that the solution is generated by a finite number of eigenvalues simply by projecting the trajectory onto the stable

[2] See Fabbri and Gozzi (2008), Freni et al. (2008), Boucekkine et al. (2010), Federico et al. (2010), Bambi et al. (2012)

eigenspace. Conditions for existence and determinacy are obtained by comparing the number of roots with negative real parts and the number of initial conditions. We will also study the case of systems that include algebraic equations and define the additional constraints that must be taken into consideration.

In Section 2, we present the kind of dynamics we are interested in. The conditions for the existence and uniqueness of solutions to systems of DDEs are proposed in Section 3, whereas those of systems of ADEs are proposed in Section 4. The conclusion is presented in Section 5.

2 Presentation of the problem

In order to fix matters, let us consider a delay differential equation (DDE, hereafter). Letting $t \in \mathbb{R}_+$ denote time, the dynamic problem can be written as follows:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\left(u-t\right) x\left(u\right), \\ x\left(\theta\right) = \bar{x}\left(\theta\right) \text{ given for } \theta \in \left[-1, 0\right], \end{cases} \tag{1}$$

where x is a variable of which initial value is given by a continuous function over the interval $[-1, 0]$, where x' denotes its derivative with respect to time and μ is a measure on $[-1, 0]$. The equation in (1) features a dynamics that depends on past variables, named as the delays, on the interval $[t-1, t]$[3]. In economics, the Johansen (1959) and Solow (1960) vintage capital models are well know examples of dynamic problem described by (1). Classical results for such dynamics are presented in Diekmann et al. (1995).

In economic models, other kind of systems may appear. We will consider three dynamics that differ from (1). First, we study algebraic equations with delay that reduce to DDEs upon (a finite number of) differentiations with respect

[3]Note that the largest delay is normalized to one even though it could be any positive real number. However, we do not consider systems with infinite delays, as their characteristic roots may not be isolated.

to time. The dynamic problem now writes:

$$\begin{cases} x(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \;\text{ given for } \theta \in [-1,0]\,. \end{cases} \tag{2}$$

The main difference with the DDE presented above comes from a discontinuity that is allowed at time $t = 0$: $x\,(0^+)$ is given but may be different from $x\,(0^-)$. Indeed, $x\,(0^+)$ is given through the algebraic equation:

$$x(0^+) = \int_{-1}^{0} d\mu\,(u)\,x\,(u)\,. \tag{3}$$

To summarize, the initial value is thus given by a continuous function over the interval $[-1, 0)$ where $x\,(0^-)$ exists, and a given value $x\,(0^+)$. In both problems (1) and (2), the variable is predetermined, and is usually called backward-looking. An example of such dynamics is given in de la Croix and Licandro (1999) and Boucekkine et al. (2002) for vintage human capital issues. We will study the latter as an illustrative example in Section 3.

The second kind of dynamics we consider allows for non predetermined variables, or equivalently forward-looking variables, of which initial value at time $t = 0$ is not given. The dynamic problem can be written in the case of a DDE as follows:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \;\text{ given for } \theta \in [-1,0). \end{cases} \tag{4}$$

The initial is now given by a function that is continuous on $[-1, 0)$ and bounded in 0. An example of such dynamics is given in d'Albis, Augeraud-Véron and Venditti (2012) or Jovanovic and Yatsenko (2012).

Finally, the third dynamics aim at considering equations with advances rather than delays. For instance, a differential equation with advances (ADE, hereafter) can be written as:

$$x'(t) = \int_{t}^{t+1} d\mu\,(u-t)\,x\,(u)\,, \tag{5}$$

ADE appears as the Euler equation of some vintage capital model studied using optimal control (Boucekkine et al. 2005) or dynamic programming (Boucekkine et al. 2010).

Depending on whether $x(0)$ is given or not, the dynamics characterize a backward-looking or a forward-looking variable. Below, we study first functional differential-algebraic systems with delays, and then those with advances.

3 Functional systems with delays

Let us consider a linear system that writes as:

$$\begin{cases} \mathbf{x}_0'(t) = \int_{t-1}^{t} d\bar{\mu}_1(u-t)\, W(u)\,, \\ \mathbf{x}_1(t) = \int_{t-1}^{t} d\bar{\mu}_2(u-t)\, W(u)\,, \\ \mathbf{y}'(t) = \int_{t-1}^{t} d\bar{\mu}_3(u-t)\, W(u)\,, \\ \mathbf{x}_i(\theta) = \bar{\mathbf{x}}_i(\theta) \text{ given for } \theta \in [-1,0] \text{ and } i = \{0,1\}\,, \\ \mathbf{y}(\theta) = \bar{\mathbf{y}}(\theta) \text{ given for } \theta \in [-1,0). \end{cases} \tag{6}$$

The details of the system list as follows: $\mathbf{x}_0 \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables of which dynamics are characterized by DDEs and $\mathbf{x}_0'$ denotes its gradient; $\mathbf{x}_1 \in \mathbb{R}^{n_1^b}$ is a vector of n_1^b backward variables characterized by a algebraic equation with delays; $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of n^f forward variables characterized by DDE and $\mathbf{y}'$ denotes its gradient. $\bar{\mathbf{x}}_i$ are continuous on $[-1,0]$ and $\bar{\mathbf{y}}(\theta)$ are continuous on $[-1,0)$ and bounded in 0. Moreover, $W = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ is a vectorial function.

We assume there exists a steady-state normalized to zero and define a solution to system (6) as a function $W(t)$ of which restriction for positive time belongs to $C([0,+\infty))$, satisfies (6) and is such that $\lim_{t\to+\infty} W(t) = 0$.

Let us notice that all the results we present below can be easily extended to study solutions that converge to a Balanced Growth Path (BGP) where all variables grow asymptotically at a given growth rate by considering the detrended variables.

Let n^+ denote the number of eigenvalues with positive real parts of the characteristic function of system (6) and s the number of independent adjoint eigenvectors of the characteristic function generated by the n^+ eigenvalues. By definition, $s \leq n^b + n_1^b + n^f$. Further:

Assumption H1. There are no eigenvalues with real parts equal to zero and eigenvalues are simple.

These restrictions are often assumed for ordinary differential equations: the absence of pure imaginary roots excludes a central manifold, simple roots imply a one dimensional Jordan block. The system (6) displays a configuration with a stable manifold of infinite dimension and an unstable manifold of dimension s. Hence, provided that $s \geq 1$, the configuration is saddle point but multiple solutions may emerge. By multiple solutions, we implicitly mean an infinity of solutions since it features a continuum of initial values for forward variables that initiate a trajectory satisfying system (6) and converging to the steady-state. Further:

Assumption H2. The stable manifold is not transverse to the (x_0, x_1) coordinates.

This second assumption implies that the projection of initial conditions on the unstable manifold encounters the stable manifold. Using it, we conclude that $s \leq \min\left\{n^+, n^f\right\}$. Then, we obtain the following result.

Theorem 1. *Let H1 and H2 prevail. There exists a solution to system (6) if $n^+ = s$ and there may be no solution if $n^+ > s$. Upon existence, a solution is unique if and only if $n^f = s$.*

Proof. Since it has been assumed that algebraic equations reduce to DDEs when differentiated a finite number of times, system (6) can be rewritten as:

$$\begin{cases} \mathbf{x}'(t) = \int_{t-1}^{t} d\mu_1\,(t-u)\,V\,(u)\,, \\ \mathbf{y}'(t) = \int_{t-1}^{t} d\mu_2\,(t-u)\,V\,(u)\,, \\ \mathbf{x}\,(\theta) = \bar{\mathbf{x}}\,(\theta) \ \text{ given for } \theta \in [-1,0]\,, \\ \mathbf{y}\,(\theta) = \bar{\mathbf{y}}\,(\theta) \ \text{ given for } \theta \in [-1,0), \end{cases} \tag{7}$$

where $\mathbf{x} \in \mathbb{R}^{n-n^f}$ is a vector of backward variables (with $n \equiv n^b + n_1^b + n^f$) and $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of forward variables, and where $V = (\mathbf{x}, \mathbf{y})$. Let us first rewrite system (7) in a compact way using the linear operator L_{-}, acting on $C([-1,0],\mathbb{R}^n)$ and defined as follows:

$$L_{-}\,(V\,(t)) = \int_0^1 d\mu\,(u)\,V\,(t-u)\,.$$

To be able to study a system like (7) that incorporates forward variables, d'Albis, Augeraud-Véron and Hupkes (2012) suggest to extend the set of initial conditions to $C([-1,0],\mathbb{R}^n) \times \mathbb{R}^n$. A solution to (7) is defined as a function $V\,(t) \in T$ where:

$$T = C\,([-1,0]\,,\mathbb{R}^n) \times \{V \in C\,([0,\infty),\mathbb{R}^n) : \|V\|_{\infty} < \infty\},$$

with initial conditions $(\bar{\mathbf{x}}\,(\theta)\,,\bar{\mathbf{y}}\,(\theta))$ defined on $C\,([-1,0]\,,\mathbb{R}^n)$ by $\bar{\mathbf{y}}\,(0) = \bar{\mathbf{y}}\,(0^-)$ and such that $V\,(t)$ satisfies (7). Let us note that the solution may be multivalued at $t = 0$, which is due to the fact that $\mathbf{y}\,(0^+)$ may be different from $\bar{\mathbf{y}}\,(0^-)$. In order to deal with a possible jump at time $t = 0$ and to be able to computed it, we modify the definition of L_{-} in such a way that L_{-} is now acting on $C([-1,0],\mathbb{R}^n) \times \mathbb{R}^n$ and is defined as follows:

$$L_{-}\,(V\,(t)\,,v) = \int_0^1 d\mu\,(u)\,V\,(t-u) + \left(\mu\left(0^+\right) - \mu\left(0^-\right)\right)v.$$

Moreover, the initial conditions $X = (\bar{\mathbf{x}}\,(\theta)\,,\bar{\mathbf{y}}\,(\theta)\,,(\mathbf{x}\,(0^+)\,,\mathbf{y}\,(0^+)))$ now belong to $C([-1,0],\mathbb{R}^n) \times \mathbb{R}^n$.

Informations concerning the local existence and multiplicity of solutions are contained in the characteristic function. Let us denote by $\Delta_{L_-}(\lambda) = \lambda I - \int_{-1}^{0} d\mu(u)\, e^{\lambda u}$, the characteristic function of (7). It can be computed as follows:

$$\Delta_{L_-}(\lambda) = \prod_{i=1}^{N_1^b} (\lambda - \alpha_i)\, \delta_{L_-}(\lambda),$$

where $\delta_{L_-}(\lambda)$ is the characteristic function of system (6) and where $(\alpha_i)_{1 \leq i \leq N_1^b}$ denote the N_1^b roots that appear as a consequence of the differentiation of the algebraic equations of system (6). If algebraic equations reduce to differential equation when differentiated once with respect to time, $N_1^b = n_1^b$. If this reduction needs more than one differentiation, $N_1^b > n_1^b$ but N_1^b conditions are now provided at time $t = 0$.

Let us denote by $Q_{\alpha_i}(X)$ the spectral projection on the vector space spanned by $e^{\alpha_i t}$. We have: $Q_{\alpha_i}(X) = e^{\alpha_i t} H_{\alpha_i} R_{\alpha_i}(X)$, where:

$$R_{\alpha_i}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{3} \int_{-1}^{0} d\bar{\mu}_j(u)\, e^{\alpha_i u} \int_{u}^{0} e^{-\alpha_i s} d\bar{\mu}_j(s)\, (\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{α_i} is a matrix such that: $\Delta_{L_-}(\alpha_i)\, H_{\alpha_i} = H_{\alpha_i} \Delta_{L_-}(\alpha_i) = 0$. The computation of $H_{\alpha_i} R_{\alpha_i}(X)$ (see Theorem 3.16 in d'Albis, Augeraud-Véron and Hupkes, 2011) permits to see that it is proportional to

$$\mathbf{x}_1(0) - \int_{-1}^{0} d\bar{\mu}_2(u)\, W(u),$$

which implies that: $H_{\alpha_i} R_{\alpha_i}(X) = 0$.

Let us assume in the following that $\Delta_{L_-}(\lambda) = 0$ has no roots with real part equal to 0 and let us denote by n^+ the number of roots with positive real parts and which are distinct to any α_i.

If $n^+ = 0$, there is no unstable manifold, which implies that the set of initial conditions leading to a solution is $C([-1, 0], \mathbb{R}^n) \times \mathbb{R}^n$. For any initial condition $(\mathbf{x}(\theta), \mathbf{y}(\theta)) \in C([-1, 0], \mathbb{R}^n)$ with $\mathbf{y}(0) = \mathbf{y}(0^-)$, and any $(\mathbf{x}(0^+), \mathbf{y}(0^+))$, a continuous and bounded solution can be found.

If $n^+ > 0$, there exists an unstable manifold and one need to use the spectral projection formula to describe the solutions to system (7). Let $(\lambda_j)_{1\leq j\leq n^+}$ be the characteristic roots with positive real part of $\delta_{L_-}(\lambda) = 0$. The spectral projection $Q_{\lambda_j}(X)$ on the vector space spanned by $e^{\lambda_j t}$ is $Q_{\lambda_j}(X) = e^{\lambda_j t} H_{\lambda_j} R_{\lambda_j}(X)$ where:

$$R_{\lambda_j}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{2} \int_{-1}^{0} d\mu_j(u)\, e^{\lambda_j u} \int_{u}^{0} e^{-\lambda_j s} d\mu_j(s)\, (\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{λ_j} is such that: $\Delta_{L_-}(\lambda_j) H_{\lambda_j} = H_{\lambda_j} \Delta_{L_-}(\lambda_j) = 0$. As the dynamics belong to the stable manifold, the projection on the unstable manifold should be null, which formally writes:

$$Q_{\lambda_j}(X) = 0. \tag{8}$$

We thus obtain a system of n^+ equations with n^f unknowns, which are given by $\mathbf{y}(0^+)$. Since eigenvectors may be linearly dependent, system (8) can be decomposed in two parts: a system of s equations with n^f unknowns, and $(n^+ - s)$ conditions on the initial known conditions $(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$, which are such that $\bar{\mathbf{x}}(0^-), \bar{\mathbf{x}}(0^+)$ and $\bar{\mathbf{y}}(0^-)$ are given. As the adjoint eigenvectors, denoted $(W_i^*)_{1\leq i\leq s}$, are linearly independent, we can write this formally as follows:

$$W_i^* \left(0, \mathbf{y}\left(0^+\right) - \bar{\mathbf{y}}\left(0^-\right)\right) = M_i\left(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)\right) \text{ for } 1 \leq i \leq s,$$

and

$$0 = M_i\left(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)\right) \text{ for } s+1 \leq i \leq n^+,$$

where $M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$ is an operator acting on the initial conditions, which is defined using the fact that the spectral projection on the unstable manifold has to be null. We notice that the first equation implies that W_i^* should not be colinear to the x axis if we want to avoid degeneracies. As the W_i^* are orthogonal to the stable manifold, the stable manifold should not be orthogonal to the x axis. If $s < n^f$, there are multiple solutions: some components of

$y(0^+)$ can be freely chosen to have a solution. If $n^+ > s$, there is no solution generically: whatever $y(0^+)$, the system of n^+ equations with n^f unknowns cannot be solved unless the initial condition happens to satisfy the conditions, which is not guaranteed. If $s = n^f$, the system for $\mathbf{y}(0^+) - \bar{\mathbf{y}}(0^-)$ has the same number of equations and unknowns, which implies, as the W_i^* are linearly independent, that upon existence the solution is unique. □

Corollary 1. *Provided that adjoint eigenvectors are linearly independent, the system (6) may have no solution if* $n^f < n^+$, *always has a unique solution if* $n^f = n^+$, *and always has multiple solutions if* $n^f > n^+$.

To establish a rule for existence and uniqueness, the proof of Theorem 1 aims at finding initial conditions for forward variables, i.e. for $\mathbf{y}(0^+)$, such that the projection of the dynamics on the unstable manifold is the null vector. In our case, the number of unknowns has the same dimension as $\mathbf{y}$. The number of forward variables is hence compared with the number of conditions obtained by setting the considered projection to zero; those conditions are linked to the number of eigenvalues with positive real parts. Conversely, as the dimensions of the stable manifold and the set of initial conditions are infinite, the information on the number of backward variables is not involved in the argument. As in finite dimentional system, multiple solutions implies indeterminacy.

Let us consider an illustrative example. Boucekkine et al. (2002) consider the dynamics of human capital, denoted $H(t)$, that is given by the following algebraic equation with delay:

$$H(t) = A \int_{t-P}^{t-T} m(t-z) H(z) dz, \tag{9}$$

where $A > 0$, $P > T > 0$, and $m(t-z)$ is a positive survival function. $H(t)$ is a backward variable and its initial condition can be written as $H(\theta) = \bar{H}(\theta)$ where $\theta \in [-P, 0)$ and $\bar{H}(.) \in C^b([-P, 0))$. Moreover, $\bar{H}(0^-)$ is allowed to be different from $H(0)$, which is given by:

$$H(0) = A \int_{-P}^{-T} m(-z) \bar{H}(z) \, dz. \tag{10}$$

Following Theorem 1, in order to study the existence and uniqueness properties of $H(t)$ for $t \geq 0$, we consider the characteristic equation $\delta(\lambda) = 0$, where $\delta(\lambda)$ is defined by

$$\delta(\lambda) = 1 - A \int_{T}^{P} m(z) e^{-\lambda z} dz. \tag{11}$$

The study of real roots is immediate, and there exists[4] a unique real real root γ, which is positive if and only if $1 < A \int_T^P m(z) \, dz$. This root represents the growth rate of the BGP. Existence and uniqueness of trajectories that converge to the BGP depends on the number of complex roots with real parts greater than γ. It can be shown that all complex roots, denoted $\lambda = p + iq$, satisfy $p < \gamma$. Suppose the contrary: as $\mathrm{Re}(\delta(p + iq)) = 0$, we would have:

$$1 = A \left| \int_T^P m(z) e^{-pz} \cos(qz) \, dz \right| < A \int_T^P m(z) e^{-pz} dz$$

which is impossible as $\delta > 0$ for $\gamma < p$. There exists a unique trajectory that converges to the BGP.

4 Functional systems with advances

Let us now study a linear system that writes:

$$\begin{cases} \mathbf{x}'(t) = \int_t^{t+1} d\bar{\mu}_1(u-t) W(u), \\ \mathbf{y}_0'(t) = \int_t^{t+1} d\bar{\mu}_2(u-t) W(u), \\ \mathbf{y}_1(t) = \int_t^{t+1} d\bar{\mu}_3(u-t) W(u), \\ \mathbf{x}(0) = \bar{\mathbf{x}}(0) \text{ given.} \end{cases} \tag{12}$$

where $\mathbf{x} \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables of which dynamics are characterized by ADEs and where $\mathbf{x}'$ denotes its gradient; where $\mathbf{y}_0 \in \mathbb{R}^{n^f}$

[4] Indeed, $\delta' > 0$, $\lim_{\lambda \to -\infty} \delta(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} \delta(\lambda) = 1$.

and $\mathbf{y}_1 \in \mathbb{R}^{n_1^f}$ are vectors of n^f and n_1^f forward variables characterized, respectively, by differential and algebraic equations with advances. Moreover, $W = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_1)$ is a vectorial function. A solution is defined as in the previous subsection.

Let n^- denote the number of eigenvalues with negative real parts of the characteristic function of system (12) and s the number of independent eigenvectors of the characteristic function generated by the n^- eigenvalues. Assuming H1 and provided that $s \geq 1$, the system (12) displays a saddle point configuration with an unstable manifold of infinite dimension and a stable manifold of dimension s. Further:

Assumption H3. The unstable manifold is not transverse to the (y_0, y_1) coordinates.

We obtain the following result.

Theorem 2. *Let H1 and H3 prevail. There exists a solution to system (12) if $n^b = s$ and there may be no solution if $n^b > s$. Upon existence, a solution is unique if and only if $n^- = s$.*

Proof. Since algebraic equations reduce to ADE when differentiated a finite number of time, system (12) can be rewritten as:

$$\begin{cases} \mathbf{x}'(t) = \int_t^{t+1} d\mu_1 (t-u) V(u), \\ \mathbf{y}'(t) = \int_t^{t+1} d\mu_2 (t-u) V(u), \\ x(0) = \bar{x}(0) \text{ given.} \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^{n^b}$ is a vector of backward variables and $\mathbf{y} \in \mathbb{R}^{n-n^b}$ a vector of forward variables (with $n \equiv n^b + n_1^b + n^f$), and where $V = (\mathbf{x}, \mathbf{y})$. Let n^- be the number of eigenvalues with negative real parts, and s be the number of linearly independent eigenvectors. Any element of the stable space can be written as:

$$V(t) = \sum_{j=0}^{n^-} \alpha_j v_j e^{\lambda_j t}, \tag{13}$$

where the $(\lambda_j)_{1\leq j\leq n^-}$ are the eigenvalues with negative real parts, the $(v_j)_{1\leq j\leq n^-}$ are the eigenvectors, and the $(\alpha_j)_{1\leq j\leq n^-}$ are the residues.

Evaluating the system (13) implies to solve a system with n^- unknowns and n^b equations. Since the eigenvectors $(v_j)_{1\leq j\leq n^-}$ may be linearly dependant, the system splits in two parts. Let us denote with $(w_j)_{1\leq j\leq s}$ the family of linearly independent eigenvectors. The first subsystem we obtain rewrites:

$$\sum_{j=0}^{n^-} \beta_j w_j = \bar{\mathbf{x}}(0),$$

which gives a system of s unknown $(\beta_j)_{1\leq j\leq s}$ and n^b constraints. And, when the $(\beta_j)_{1\leq j\leq s}$ are defined, we obtain a second system that rewrites:

$$\sum_{j=0}^{n^-} \alpha_j v_j = \sum_{j=0}^{s} \beta_j w_j,$$

which leads to a system of s equations and n^- unknowns, namely the $(\alpha_j)_{1\leq j\leq n^-}$.

□

Corollary 2. *Provided that eigenvectors are linearly independent, the system (12) may have no solution if $n^- < n^b$, always has a unique solution if $n^- = n^b$, and always has multiple solutions if $n^- > n^b$.*

We see that the rule that permits to establish the existence and uniqueness of solutions is different from the one presented in Theorem 1. With advances, as the dimension of the unstable manifold is infinite, the idea is to find initial conditions for forward variables that permit to write the dynamics on the stable manifold. This is why we use the number of eigenvalues with negative real parts to state whether the solution exists and is unique. Since we rewrite the system as a finite dimensional system, the proof of Theorem 2 is similar to what can be found for ordinary differential equations.

5 Conclusion

This paper proposes theorems for the existence and uniqueness of solutions to systems of differential or algebraic equations with delays or advances. These theorems propose conditions that link the space of unknown initial conditions to the sign of the roots of the characteristic equation, just like the well-known Blanchard-Kahn conditions. They could therefore contribute to the development of the use of DDEs and ADEs in economics, which would enable the analytical study of many phenomena. Sometimes, certain economic dynamics are characterized by differential equations that have both delays and advances. In such cases, both the stable and unstable manifolds are of infinite dimensions. Therefore, the existence and uniqueness of their solution cannot be analyzed using the theorems developed in this paper. This problem has been noted for further study.

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